

SECOND ORDER DIFFERENCE EQUATIONS AND DISCRETE ORTHOGONAL POLYNOMIALS OF TWO VARIABLES

YUAN XU

ABSTRACT. The second order partial difference equation of two variables

$$\begin{aligned} \mathcal{D}u := & A_{1,1}(x)\Delta_1\nabla_1u + A_{1,2}(x)\Delta_1\nabla_2u + A_{2,1}(x)\Delta_2\nabla_1u + A_{2,2}(x)\Delta_2\nabla_2u \\ & + B_1(x)\Delta_1u + B_2(x)\Delta_2u = \lambda u, \end{aligned}$$

is studied to determine when it has orthogonal polynomials as solutions. We derive conditions on \mathcal{D} so that a weight function W exists for which $W\mathcal{D}u$ is self-adjoint and the difference equation has polynomial solutions which are orthogonal with respect to W . The solutions are essentially the classical discrete orthogonal polynomials of two variables.

1. INTRODUCTION

Let Δ and ∇ denote the forward and the backward difference operators, respectively,

$$\Delta f(t) = f(t+1) - f(t) \quad \text{and} \quad \nabla f(t) = f(t) - f(t-1).$$

The second order difference equation in one variable takes the form

$$(1.1) \quad a(t)\Delta\nabla u + b(t)\Delta u = \lambda u, \quad \lambda \in \mathbb{R}.$$

In the case that a and b are polynomials, it is known that some of the difference equations can have orthogonal polynomials as solutions. Such difference equations have been characterized completely and the orthogonal polynomials so derived are the classical discrete orthogonal polynomials (Hahn, Meixner, Krawtchouk and Charlier polynomials), see [1] for a survey in this direction and for references.

The purpose of the present paper is to consider the second order difference equations in two variables. Throughout this paper we use the notation

$$e_1 = (1, 0) \quad \text{and} \quad e_2 = (0, 1).$$

For $x = (x_1, x_2) \in \mathbb{R}^2$ we define the forward and the backward partial difference operators by

$$(1.2) \quad \Delta_i u(x) = f(x + e_i) - f(x) \quad \text{and} \quad \nabla_i u(x) = f(x) - f(x - e_i), \quad i = 1, 2,$$

respectively. We consider the second order partial difference equation

$$(1.3) \quad \begin{aligned} \mathcal{D}u := & A_{1,1}(x)\Delta_1\nabla_1u + A_{1,2}(x)\Delta_1\nabla_2u + A_{2,1}(x)\Delta_2\nabla_1u + A_{2,2}(x)\Delta_2\nabla_2u \\ & + B_1(x)\Delta_1u + B_2(x)\Delta_2u = \lambda u, \end{aligned}$$

Date: February 1, 2008.

1991 Mathematics Subject Classification. 42C05, 33C45.

Key words and phrases. Discrete orthogonal polynomials, two variables, second order difference equation.

Work partially supported by the National Science Foundation under Grant DMS-0201669.

where $A_{i,j}$ and B_i are polynomials and λ is a real number. The goal is to characterize those equations that have orthogonal polynomials as solutions.

The orthogonal polynomials that satisfy (1.3) will be discrete orthogonal polynomials of two variables. A sequence of polynomials, $\{p_\alpha\}$, of d variables is called a sequence of discrete orthogonal polynomials if there exist a set of isolated points $V \subset \mathbb{R}^d$ and a weight function W defined on V such that p_α are orthogonal with respect to the bilinear form

$$\langle f, g \rangle = \sum_{x \in V} f(x)g(x)W(x).$$

Since we will discuss orthogonal polynomials associated with the difference equation (1.3), the set V will be a lattice set, that is, a subset of $\mathbb{Z} \times \mathbb{Z}$. We should mention that a simple change of variables $x \rightarrow hx$ shows that we can replace the difference operators Δ_i by the divided difference

$$[f(x_1 + h, x_2) - f(x_1, x_2)]/h \quad \text{and} \quad [f(x_1, x_2 + k) - f(x_1, x_2)]/k.$$

In this case the second difference $\Delta_i \nabla_j$ becomes second order divided difference; for example, $\Delta_1 \nabla_1$ and $\Delta_1 \nabla_2$ become

$$[f(x_1 + h, x_2) - 2f(x_1, x_2) + f(x_1 - h, x_2)]/h^2 \quad \text{and} \\ [f(x_1 + h, x_2) - f(x_1 + h, x_2 - k) - f(x_1, x_2) + f(x_1, x_2 - k)]/hk,$$

respectively. In other words, the difference equations (1.3) is the model for the uniform grid.

After the dilation indicated above, letting $(h, k) \rightarrow (0, 0)$, the divided difference operators become the differential operators; in particular, Δ_i becomes ∂/∂_i and $\Delta_i \nabla_j$ becomes $\partial^2/\partial_i \partial_j$. This way the difference equation becomes a differential equation. In this connection, it should be pointed out that the classification of the second order partial differential equations that have orthogonal polynomial as solutions was carried out in [8], see [2, 5, 6, 9, 12] for further work in this direction.

Comparing to the theory in one variable, the structure of discrete orthogonal polynomials in several variables is much more complicated. Some basic results are obtained in [15]; the relevant ones will be recalled in the following section.

In the next section we will also discuss for what $A_{i,j}$ and B_i the equation (1.3) has polynomial solutions. Such equations are called admissible. In Section 3 we define and discuss the self-adjoint form of the second order difference equations and give necessary and sufficient conditions for the existence of a function W so that $W\mathcal{D}$ is self-adjoint. We also define a notion of W being consistent with the difference equation, and show that consistence and self-adjointness will imply that the difference equation has polynomial solutions that are orthogonal with respect to W . In Section 4, we identify difference equations that have discrete orthogonal polynomials as solutions. Up to the mild restriction that we put on the difference equations, we believe that these are the only such difference equations. The polynomials are the classical discrete orthogonal polynomials of two variables (see, for example, [4, 13, 14]) on the set $V = \{x : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq N\}$ and on the set $V = \{x : 0 \leq x_1 \leq M, 0 \leq x_2 \leq N\}$, where both N and M can be infinity.

2. PRELIMINARY AND ADMISSIBLE EQUATIONS

2.1. Discrete orthogonal polynomials. Let V be a subset of \mathbb{R}^d . Orthogonal polynomials on V depend on the structure of the polynomial ideal $I(V) :=$

$\{p \in \mathbb{R}[x] : p(x) = 0, \forall x \in V\}$ that has V as its variety. The discrete orthogonal polynomials on V can only consist of polynomials that do not belong to $I(V)$. Such polynomial subspaces have been identified and the basic structure of discrete orthogonal polynomials of several variables, including three-term relation and Favard's theorem, has been studied in [15].

Let $\mathbb{R}[V]$ denote the subspace of polynomials. This is the space that orthogonal polynomials belong to. Then $\mathbb{R}[V] \cong \mathbb{R}[x_1, \dots, x_d]/I(V)$. There is a lattice set $\Lambda = \Lambda(V)$ such that every polynomial $P \in \mathbb{R}[V]$ can be written as

$$P(x) = \sum_{\alpha \in \Lambda} c_\alpha x^\alpha \pmod{I(V)}, \quad c_\alpha \in \mathbb{R}.$$

One particular result in [15] shows that the set Λ satisfies the following property

$$\alpha \in \Lambda \text{ implies } \alpha - \beta \in \Lambda, \text{ whenever } \alpha - \beta \in \mathbb{N}_0^d \text{ and } \beta \in \mathbb{N}_0^d.$$

For two variables, $d = 2$, this means that Λ must be of a stair shape; more precisely, there is a sequence of positive integers n_i , which satisfies $n_m \leq n_{m-1} \leq \dots \leq n_0$ (some of the n_i can be positive infinity and so can m), such that

$$(2.1) \quad \Lambda = \{(k, l) : 0 \leq l \leq m, 0 \leq k \leq n_l\}.$$

Hence, any polynomial $P \in \mathbb{R}[V]$ can be written as

$$P(x) = \sum_{l=0}^m \sum_{j=0}^{n_l} c_{l,j} x_1^l x_2^j, \quad c_{l,j} \in \mathbb{R}.$$

We will assume that Λ contains $\{(i, j) : 0 \leq i + j \leq 2\}$ and at least one of the points $(2, 1)$ and $(1, 2)$.

Given Λ as in (2.1), the highest degree of monomials appeared in $\mathbb{R}[V]$ is denoted by \mathbf{n} , which could be positive infinity. It follows that

$$\mathbf{n} := \max\{n_l + l : 0 \leq l \leq m\}.$$

For $\Lambda = \Lambda(V)$, denote $\Lambda_k(V) = \{(i, j) : i + j = k\}$ for each $k < \mathbf{n}$. Let r_k denote the distinct elements in $\Lambda_k(V)$, which is the number of monomials of degree exactly k in $\mathbb{R}[V]$. For each j satisfying $0 \leq j \leq \mathbf{n}$, it can be verified that

$$(2.2) \quad r_k = k + 1 - \sum_{l=0}^m (k - l - n_l)_+,$$

where $(a)_+ = 0$ if $a \leq 0$ and $(a)_+ = a$ if $a > 0$.

Let W be a real function on V and $W(x) \neq 0$ on V for any $x \in V$. Assume that

$$\sum_{x \in V} |x_1^i x_2^j| |W(x)| < \infty \quad \text{for all } (i, j) \in \mathbb{N}_0^2$$

in the case V is an infinite set. Let $\Pi^2 = \mathbb{R}[x_1, x_2]$ and let Π_n^2 denote the subspace of polynomials of degree at most n . Define the bilinear form $\langle \cdot, \cdot \rangle$ on $\Pi^2 \times \Pi^2$ by

$$\langle f, g \rangle = \mathcal{L}(fg), \quad \text{where } \mathcal{L}(f) := \sum_{x \in V} f(x)W(x).$$

If $\langle f, g \rangle = 0$, we say that f and g are orthogonal to each other with respect to W on the discrete set V . A polynomial of degree k is said to be an orthogonal polynomial with respect to W if it is orthogonal to all polynomials of degree lower than k in $\mathbb{R}[V]$. For a given weight function W , let \mathcal{V}_k denote the space of orthogonal polynomials of degree exactly k on V . The dimension of \mathcal{V}_k is r_k in (2.2). Only in

the case of $V = \{x : x_1 \geq 0, x_2 \geq 0, 0 \leq x_1 + x_2 \leq N\}$ this dimension is $r_k = k + 1$, which is equal to the dimension for the space of continuous orthogonal polynomials (see [3]).

Since we are interested in discrete orthogonal polynomials that are solutions of the difference equation (1.3), the set V is a lattice set. If u satisfies the difference equation (1.3) then the function $u^* = u(x_1 - k, x_2 - l)$ will satisfy a similar difference equation. Hence we can assume that $(0, 0)$ belong to V . For a given $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, let $B_{i,j}$ denote the square that has $(0, 0), (i, 0), (0, j)$ and (i, j) as corners. In order that the differences make sense, we further assume that V satisfies the following property:

$$(2.3) \quad (i, j) \in V \quad \text{implies} \quad (k, l) \in V, \quad \text{whenever } (k, l) \in B_{i,j}.$$

As pointed out in [15], in some cases, the index set Λ can be the same as the set V . This happens, in particular, in the following two important cases,

Type A. $V = \{(x_1, x_2) : 0 \leq x_1 \leq M, 0 \leq x_2 \leq N\}$,

Type B. $V = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, 0 \leq x_1 + x_2 \leq N\}$,

that are of particular interest to us. We note that both N and M can be infinity.

2.2. Admissible difference equations. Since the forward and the backward difference equations are related by

$$(2.4) \quad \Delta_j = \nabla_j \Delta_j + \nabla_j,$$

a second order partial difference operator can be expressed in different forms. We shall fix one particular form and call it the standard form.

Definition 2.1. Let $A_{i,j}$ and B_i , $i, j = 1, 2$, be polynomials of two variables. We call the operator \mathcal{D} defined by

$$(2.5) \quad \begin{aligned} \mathcal{D} := & A_{1,1}(x) \Delta_1 \nabla_1 + A_{1,2}(x) \Delta_1 \nabla_2 + A_{2,1}(x) \Delta_2 \nabla_1 + A_{2,2}(x) \Delta_2 \nabla_2 \\ & + B_1(x) \Delta_1 + B_2(x) \Delta_2 \end{aligned}$$

a standard second order partial difference operator.

An alternative way of writing a second order partial difference operator is

$$\begin{aligned} \tilde{\mathcal{D}} := & \tilde{A}_{1,1}(x) \Delta_1 \nabla_1 + \tilde{A}_{1,2}(x) \Delta_1 \nabla_2 + \tilde{A}_{2,1}(x) \Delta_2 \nabla_1 + \tilde{A}_{2,2}(x) \Delta_2 \nabla_2 \\ & + \tilde{B}_1(x) \nabla_1 + \tilde{B}_2(x) \nabla_2, \end{aligned}$$

which can be easily converted to the standard form by using (2.4).

Definition 2.2. The equation $\mathcal{D}u = \lambda u$ is called admissible on V if for any $k \in \mathbb{N}_0$, there is a number λ_k such that the equation $\mathcal{D}u = \lambda_k u$ has r_k linearly independent polynomial solutions and it has no non-trivial solutions in the set of polynomials of degree small than k .

The following proposition shows that we can assume that the linearly independent polynomial solutions are given by those polynomials whose highest term contains a single monomial.

Proposition 2.3. The equation $\mathcal{D}u = \lambda u$ is admissible in $\mathbb{R}[V]$ if and only if, for every integer k that satisfies $0 \leq k \leq n$, there exists a number λ_k such that the equation $\mathcal{D}u = \lambda_k u$ has r_k linearly independent solutions in the form of

$$(2.6) \quad P_{k,l}(x) = x_1^k x_2^l + R_{k,l}(x), \quad (k, l) \in \Lambda_k,$$

where $R_{k,l} \in \Pi_k^2$, and the equation has no non-trivial solutions in $\mathbb{R}[\Lambda]$ of degree less than k .

Proof. That the equation is admissible implies that, for each k , there exist solutions $Q_{i,j} \in \mathbb{R}[V]$, $\deg Q_{i,j} = k$. Introducing the notation

$$x_\Lambda^k = \{x_1^i x_2^j : (i, j) \in \Lambda, \quad i + j \leq k\}$$

and regard it as a column vector, we can write

$$Q_{i,j}(x) = a_{i,j}^T x_\Lambda^k + R_{i,j}, \quad \deg R_{i,j} < k$$

where $a_{i,j}^T$ is a row vector of real numbers. We show that $\{a_{i,j}^T x_\Lambda^k : (i, j) \in \Lambda\}$ is linearly independent. Indeed, suppose otherwise, then there exist $c_{i,j}$ such that $\sum_{(i,j) \in \Lambda} c_{i,j} a_{i,j}^T x_\Lambda^k = 0$, which implies that

$$\sum_{(i,j) \in \Lambda} c_{i,j} Q_{i,j} = \sum_{(i,j) \in \Lambda} c_{i,j} R_{i,j} := R_k \in \Pi_{k-1}^2.$$

Since $Q_{i,j}$ satisfies $\mathcal{D}u = \lambda_k u$, we conclude that

$$\mathcal{D}R_k = \mathcal{D} \sum_{(i,j) \in \Lambda} c_{i,j} Q_{i,j} = \lambda_k \sum_{(i,j) \in \Lambda} c_{i,j} Q_{i,j} = \lambda_k R_k$$

so that $R_k \equiv 0$ by the definition of admissibility. Consequently, since the set of $\{Q_{i,j}\}$ is linearly independent, $c_{i,j} = 0$ for $(i, j) \in \Lambda$. Thus, $\{a_{i,j}^T x_\Lambda^k : (i, j) \in \Lambda_k\}$ is linearly independent and the matrix $(a_{i,j}^T)_{(i,j) \in \Lambda_k}$ is invertible. Let

$$(P_{i,j})_{(i,j) \in \Lambda_k} := [(a_{i,j}^T)_{(i,j) \in \Lambda_k}]^{-1} (Q_{i,j})_{(i,j) \in \Lambda_k};$$

then $P_{i,j}$ are polynomials of the form $P_{i,j}(x) = x_1^i x_2^j + p_{i,j}$ where $\deg p_{i,j} < i + j = k$. This proves one direction of the proposition. The other direction is trivial, since the set $\{P_{i,j}\}$ in (2.6) is evidently linear independent. \square

Proposition 2.4. *For the equation $\mathcal{D}u = \lambda u$ to be admissible it is necessary that $A_{i,j}$ are polynomials of degree at most 2 and B_i are polynomials of degree at most 1.*

Proof. Suppose that $\mathcal{D}u = \lambda u$ is admissible. Then there is a system of polynomials, $P_{i,j}$ as in (2.6), which are solutions of the equation. Let $A_{i,j}^0$ and B_i^0 denote the part of $A_{i,j}$ and B_i that has degree larger than 2, respectively. We need to prove that $A_{i,j}^0(x) \equiv 0$ and $B_i^0(x) \equiv 0$. Substituting $P_{i,j}$ into $\mathcal{D}u = \lambda u$ and examining the highest terms of the resulting equation, we obtain

$$\begin{aligned} & A_{1,1}^0(x) \Delta_1 \nabla_1 x_1^i x_2^j + A_{1,2}^0(x) \Delta_1 \nabla_2 x_1^i x_2^j + A_{2,1}^0(x) \Delta_2 \nabla_1 x_1^i x_2^j \\ & + A_{2,2}^0(x) \Delta_2 \nabla_2 x_1^i x_2^j + B_1^0(x) \Delta_1 x_1^i x_2^j + B_2^0(x) \Delta_2 x_1^i x_2^j = 0 \end{aligned}$$

for $(i, j) \in \Lambda$. Let $(i, j) = (1, 0)$ and $(i, j) = (0, 1)$; then $\Delta_i \nabla_j x_1^i x_2^j = 0$ and we conclude that $B_1^0(x) = B_2^0(x) \equiv 0$. Next we let $(i, j) = (2, 0)$; then only $\Delta_1 \nabla_1 x_1^i x_2^j$ is nonzero and, consequently, $A_{1,1}^0(x) \equiv 0$. Similarly, we deduce $A_{2,2}^0(x) \equiv 0$ by setting $(i, j) = (0, 2)$. Furthermore, the choice $(i, j) = (1, 1)$ leads to

$$A_{1,2}^0(x) + A_{2,1}^0(x) \equiv 0.$$

Finally, let $(i, j) = (1, 2)$ (or $(2, 1)$); we conclude that

$$(2x_1 + 1)A_{1,2}^0(x) + (2x_1 - 1)A_{2,1}^0(x) \equiv 0.$$

Together, the last two displayed equations imply $A_{1,2}^0(x) = A_{2,1}^0(x) \equiv 0$. \square

Since $A_{i,j}$ are polynomials of second degree and B_i are polynomials of the first degree, we can assume that they have the form.

$$\begin{aligned} A_{i,j}(x) &= a_{i,j}^{2,0}x_1^2 + a_{i,j}^{1,1}x_1x_2 + a_{i,j}^{0,2}x_2^2 + a_{i,j}^{1,0}x_1 + a_{i,j}^{0,1}x_2 + a_{i,j}^{0,0}; \\ B_i(x) &= b_i^1x_1 + b_i^2x_2 + b_i^0, \quad i, j = 1, 2. \end{aligned}$$

Theorem 2.5. *Assume that the quadratic parts of $A_{1,2}$ and $A_{2,1}$ are equal; that is, $A_{1,2} - A_{2,1} \in \Pi_1^2$. Then the equation $\mathcal{D}u = \lambda u$ is admissible if and only if*

$$(2.7) \quad A_{i,j}(x) = ax_ix_j + a_{i,j}x_1 + b_{i,j}x_2 + c_{i,j} \quad \text{and} \quad B_i = bx_i + d_i$$

for $i, j = 1, 2$ with $c_{1,1} + c_{1,2} + c_{2,1} + c_{2,2} = 0$ and

$$(2.8) \quad \lambda_k = k(ka - a + b), \quad 0 \leq k \leq \mathbf{n},$$

where a and b are real numbers such that $ap + b \neq 0$ for all nonnegative integer p satisfying $0 \leq p \leq \mathbf{n}$.

Proof. Again consider the highest terms of the polynomials in $\mathcal{D}P_{k,l} - \lambda P_{k,l}$. Since $\Delta_i x_i^j = (x_i + 1)^j - x_i^j = jx_i^{j-1} + \dots$ and $\nabla_i x_i^j = x_i^j - (x_i - 1)^j = jx_i^{j-1} + \dots$, we obtain

$$\begin{aligned} &A_{1,1}(x)k(k-1)x_1^{k-2}x_2^l + A_{1,2}(x)klx_1^{k-1}x_2^{l-1} + A_{2,1}(x)klx_1^{k-1}x_2^{l-1} \\ &+ l(l-1)A_{2,2}(x)x_1^kx_2^{l-2} + B_1(x)kx_1^{k-1}x_2^l + B_2(x)lx_1^kx_2^{l-1} = \lambda_{j+k}x_1^kx_2^l. \end{aligned}$$

Using the formula of $A_{i,j}$ and B_i , we can then derive equations on the coefficients of $A_{i,j}$ and B_i by comparing the coefficients of $x_1^kx_2^l$.

The coefficients of $x_1^{k-2}x_2^{l+2}$ and $x_1^{k+2}x_2^{l-2}$ lead to equations

$$k(k-1)a_{1,1}^{0,2} = 0 \quad \text{and} \quad l(l-1)a_{2,2}^{2,0} = 0$$

which imply that $a_{1,1}^{0,2} = a_{2,2}^{2,0} = 0$. Using the assumption that $a_{1,2}^{k,l} = a_{2,1}^{k,l}$ for $k+l=2$, the coefficients for $x_1^{k-1}x_2^{l+1}$, $x_1^kx_2^l$ and $x_1^{k+1}x_2^{l-1}$ lead to the equations

$$\begin{aligned} &k(k-1)a_{1,1}^{1,1} + 2kla_{1,2}^{0,2} + kb_1^2 = 0, \\ (2.9) \quad &k(k-1)a_{1,1}^{2,0} + 2kla_{1,2}^{1,1} + l(l-1)a_{2,1}^{1,1} + kb_1^1 + lb_2^2 = \lambda_{k+l}, \\ &2kla_{1,2}^{2,0} + l(l-1)a_{2,2}^{1,1} + lb_2^1 = 0 \end{aligned}$$

for $(k, l) \in \Lambda$. In particular, if $(k, l) = (1, 0)$ and $(k, l) = (0, 1)$ then the first and the third equations of (2.9) show that $b_1^2 = 0$ and $b_2^1 = 0$, respectively. Furthermore, $k=1$ in the first equation shows that $a_{1,2}^{0,2} = 0$ and, consequently, $a_{1,1}^{1,1} = 0$. Similarly, $l=1$ in the third equation shows that $a_{1,2}^{2,0} = 0$ and $a_{2,2}^{1,1} = 0$. Setting $k=0$ or $l=0$ in the second equation of (2.9) leads to the equations

$$l(l-1)a_{2,2}^{0,2} + lb_2^2 = \lambda_l \quad \text{and} \quad k(k-1)a_{1,1}^{2,0} + kb_1^1 = \lambda_k.$$

In particular, $k=l=1$ gives $b_2^2 = b_1^1$ and, consequently, $a_{1,1}^{2,0} = a_{2,2}^{0,2}$. Furthermore, letting $(k, l) = (1, 1)$ and $(k, l) = (2, 0)$ in the second equation of (2.9) gives the equation

$$2a_{1,2}^{1,1} + 2b = \lambda_2 \quad \text{and} \quad 2a_{1,1}^{2,0} + 2b = \lambda_2$$

where we write $b = b_1^2 = b_2^1$. Consequently, we conclude that $a_{1,1}^{2,0} = a_{1,2}^{1,1}$. Writing $a = a_{1,1}^{2,0} = a_{1,2}^{1,1} = a_{2,2}^{0,2}$, we have proved that $A_{i,j}$ and B_i are in the forms specified. Moreover, the second equation of (2.9) with $l = 0$ or with $k = 0$ shows that

$$k(k-1)a + kb = \lambda_k \quad \text{and} \quad l(l-1)a + lb = \lambda_l.$$

This is (2.8) and that $\lambda_l \neq 0$ shows that $ap + b \neq 0$ for all nonnegative integer p that satisfies $0 \leq p \leq \mathbf{n} - 1$.

On the other hand, assume that $A_{i,j}$ and B_i are given as in (2.7) and λ_k is given as in (2.8). Let $(t)_k = t(t+1) \dots (t+k-1)$ be the Pochhammer symbol. Define

$$m_k(t) = (t)_k/k! \quad \text{and} \quad m_{k,l}(x) = m_k(x_1)m_l(x_2).$$

For each pair of nonnegative integers (r, s) satisfying $r + s = n$, substituting the polynomial

$$P_{r,s}(x) = m_{r,s}(x) + \sum_{k+l < n} f_{k,l} m_{k,l}(x)$$

into the difference equation $\mathcal{D}u = \lambda_n u$ leads to a linear system of equations for the coefficients $f_{i,j}$. Indeed, the definition of $m_{k,l}$ implies that

$$\Delta m_k(x) = m_{k-1}(x+1) = \sum_{j=0}^{k-1} m_{j,l}(x) \quad \text{and} \quad \nabla m_k(x) = m_{k-1}(x),$$

from which it follows that

$$\Delta_1 \nabla_1 m_{k,l} = \sum_{j=0}^{k-2} m_{j,l}, \quad \text{and} \quad \Delta_1 \nabla_2 m_{k,l} = \sum_{j=0}^{k-1} m_{j,l-1},$$

and similar formula for $\nabla_1 \Delta_2 m_{k,l}$ and $\Delta_2 \nabla_2 m_{k,l}$. Furthermore, there is also $t m_k(t) = (k+1)m_{k+1}(t) - k m_k(t)$, from which follows formulas such as

$$\begin{aligned} x_1^2 \Delta_1 \nabla_1 m_{k,l}(x) &= k(k-1)m_{k,l} - (k-1)^2 m_{k-1,l} \\ x_1 x_2 \Delta_1 \nabla_1 m_{k,l}(x) &= k l m_{k,l}(x) - k(l-1)m_{k,l-1}(x). \end{aligned}$$

Using these formulas, a tedious computation shows that the coefficients of $m_{r,s}$ in the expression $\mathcal{D}P_{r,s} - \lambda_n P_{r,s}$ are linear in $f_{k,l}$ with $r+s \leq k+l \leq m$. Since the set of $m_{k,l}$ is clearly linearly independent, these coefficients must be zero, which gives a linear system of equations in variables $f_{k,l}$.

The coefficient of $m_{r,s}$ turns out to be $c_{1,1} + c_{1,2} + c_{2,1} + c_{2,2}$, which is zero by assumption. Clearly, this also shows that this condition is necessary. The coefficients of $m_{r-1,s}$ and $m_{r,s-1}$ lead to the following two equations

$$\begin{aligned} (r-1)(-a(r-1) + a_{1,1}) + s(-a(r-1) + a_{2,1}) + s b_{1,2} + d_1 + \lambda_{r-1,s} f_{r-1,s} &= 0, \\ r(-a(s-1) + a_{1,2}) + (s-1)(-a(s-1) + a_{2,2}) + r b_{2,1} + d_2 + \lambda_{r,s-1} f_{r,s-1} &= 0, \end{aligned}$$

from which $f_{r-1,s}$ and $f_{r,s-1}$ are determined uniquely. The coefficient of $m_{k,l}$ for $k+l < r+s$ is a linear combination of $f_{p,q}$ in which $p+q \geq k+l$ and $f_{k,l}$ is the only element with $p+q = k+l$. The coefficient of $f_{k,l}$ is

$$\lambda_{k,l} - \lambda_{r,s} = (k+l-r-s)(b+a(-1+k+l+r+s))$$

which is nonzero by assumption. Hence, starting from $f_{r,s-1}$ and $f_{r-1,s}$ and works our way down, we conclude that $f_{k,l}$ is uniquely determined. \square

If u satisfies an admissible difference equation $\mathcal{D}u$, then under a change of variables $x \mapsto x + h$, where $h \in \mathbb{Z}^2$, the function $\tilde{u}(x) = u(x + h)$ will satisfy a similar admissible difference equation. We consider these equations equivalent. In particular, we can choose h such that $B_1(x) = bx_1$ and $B_2(x) = bx_2$.

We note that an affine transform $x_1 = p_{1,1}x'_1 + p_{1,2}x'_2$, $x_2 = p_{2,1}x'_1 + p_{2,2}x'_2$, however, will alter the difference equations significantly.

The characterization of partial differential equations in two variables that have orthogonal polynomials as solutions is given up to affine transformations; in other words, if $u(x)$ satisfies a differential equation, then the function $u^*(x') := u(x)$ under the affine transformation also satisfies a similar differential equation and the two equations are considered to be the same type. This allows us to use affine transformation to reduce each type of equations to its simplest form and proceed from there. For the difference equations, however, the affine transformation can no longer be used as a tool to reduce the type of equations.

3. SELF-ADJOINT DIFFERENCE EQUATION AND THE WEIGHT FUNCTION

3.1. Self-adjoint difference equations. For the difference equation in one variable, the self-adjoint form is

$$\Delta(\sigma(x)\nabla u(x)) - \lambda u(x) = 0.$$

For difference equations in two variables, we introduce the following definition:

Definition 3.1. Let $\sigma_{i,j}$ be continuous functions of two variables. The operator

$$\mathcal{E}u = \nabla_1 [\sigma_{1,1}(x)\Delta_1 u + \sigma_{1,2}(x)\Delta_2 u] + \nabla_2 [\sigma_{2,1}(x)\Delta_1 u + \sigma_{2,2}(x)\Delta_2 u]$$

is called a self-adjoint second order difference operator. The equation $\mathcal{E}u = \lambda u$ is called a self-adjoint difference equation.

Proposition 3.2. Let \mathcal{D} be the difference operator in (2.5). There is a function W such that $W(x)\mathcal{D}$ is self-adjoint if and only if W satisfies

$$(3.1) \quad \begin{aligned} WB_1 &= \Delta_1(WA_{1,1}) + \Delta_2(WA_{1,2}) \\ WB_2 &= \Delta_1(WA_{2,1}) + \Delta_2(WA_{2,2}). \end{aligned}$$

Proof. First we rewrite the self-adjoint form of $\mathcal{E}u$ in the standard form of the difference operator $\mathcal{D}u$. For this purpose the following formulas are useful:

$$(3.2) \quad \Delta_i(fg) = f\Delta_i g + g(\cdot + e_i)\Delta_i f \quad \text{and} \quad \nabla_i(fg) = f\nabla_i g + g(\cdot - e_i)\nabla_i f$$

where $i = 1, 2$ and

$$(3.3) \quad \nabla_i u(\cdot + e_j) = \Delta_j \nabla_i u + \nabla_i u = \Delta_j \nabla_i u + \Delta_i u - \Delta_i \nabla_i u, \quad i, j = 1, 2;$$

all can be easily verified. Using these formulas we can write

$$\begin{aligned} \mathcal{E}u &= \sigma_{1,1}(\cdot - e_1)\Delta_1 \nabla_1 u + \sigma_{1,2}(\cdot - e_2)\Delta_1 \nabla_2 u \\ &\quad + \sigma_{2,1}(\cdot - e_1)\Delta_2 \nabla_1 u + \sigma_{2,2}(\cdot - e_2)\Delta_2 \nabla_2 u \\ &\quad + (\nabla_1 \sigma_{1,1} + \nabla_2 \sigma_{1,2})\Delta_1 u + (\nabla_1 \sigma_{2,1} + \nabla_2 \sigma_{2,2})\Delta_2 u. \end{aligned}$$

Comparing with the standard form $\mathcal{D}u$ in (2.5) and solving for $\sigma_{i,j}$ leads to

$$\begin{aligned} A_{1,1} &= \sigma_{1,1}(\cdot - e_1), \quad A_{1,2} = \sigma_{1,2}(\cdot - e_2), \quad B_1 = \nabla \sigma_{1,1} + \nabla_2 \sigma_{1,2}, \\ A_{2,1} &= \sigma_{2,1}(\cdot - e_1), \quad A_{2,2} = \sigma_{2,2}(\cdot - e_2), \quad B_2 = \nabla_1 \sigma_{1,2} + \nabla_2 \sigma_{2,2}. \end{aligned}$$

Consequently, it follows that

$$\sigma_{i,j} = A_{i,j}(\cdot + e_j), \quad 1 \leq i, j \leq 2.$$

Using the fact that $\nabla_i u(\cdot + e_i) = \Delta_i u$, we then obtain

$$B_1 = \Delta_1 A_{1,1} + \Delta_2 A_{1,2} \quad \text{and} \quad B_2 = \Delta_1 A_{2,1} + \Delta_2 A_{2,2}.$$

That is, the operator \mathcal{D} is self-adjoint only if the above relations hold. Evidently, if B_1 and B_2 are given by the above relation, then $\mathcal{D}u$ can be written in the self-adjoint form. Hence, \mathcal{D} is self adjoint if and only if these relations hold. Consequently, $W\mathcal{D}$ is self-adjoint if and only if (3.1) holds. \square

Using (3.2), the system of equations (3.1) can be written as

$$(3.4) \quad \begin{aligned} A_{1,1}(x + e_1)W(x + e_1) + A_{1,2}(x + e_2)W(x + e_2) &= r_1(x) \\ A_{2,1}(x + e_1)W(x + e_1) + A_{2,2}(x + e_2)W(x + e_2) &= r_2(x) \end{aligned}$$

where r_i are polynomials defined by

$$r_1 = A_{1,1} + A_{1,2} + B_1 \quad \text{and} \quad r_2 = A_{2,1} + A_{2,2} + B_2.$$

If the determinant of this system of equations is nonzero, then we can solve for $\nabla_1 W$ and $\nabla_2 W$. Let us define

$$\begin{aligned} \alpha(x) &:= A_{1,1}(x + e_1)A_{2,2}(x + e_2) - A_{1,2}(x + e_2)A_{2,1}(x + e_1), \\ \beta_1(x) &:= r_1(x)A_{2,2}(x + e_2) - r_2(x)A_{1,2}(x + e_2), \\ \beta_2(x) &:= r_2(x)A_{1,1}(x + e_1) - r_1(x)A_{2,1}(x + e_1). \end{aligned}$$

Theorem 3.3. *Assume that $\alpha(x) \neq 0$ for $x \in V$. Then there is a W such that $W\mathcal{D}u$ is self-adjoint if and only if*

$$(3.5) \quad \beta_1(x)\alpha(x + e_2)\beta_2(x + e_1) = \beta_2(x)\alpha(x + e_1)\beta_1(x + e_2), \quad x \in V.$$

Proof. Since $\alpha(x) \neq 0$ means the determinant of the system of equations (3.4) is nonzero, we can solve (3.4) to get

$$(3.6) \quad \frac{W(x + e_1)}{W(x)} = \frac{\beta_1(x)}{\alpha(x)} \quad \text{and} \quad \frac{W(x + e_2)}{W(x)} = \frac{\beta_2(x)}{\alpha(x)}.$$

These equations imply

$$\begin{aligned} W(x + e_1 + e_2) &= W(x + e_1) \frac{\beta_2(x + e_1)}{\alpha(x + e_1)} = W(x) \frac{\beta_1(x)}{\alpha(x)} \frac{\beta_2(x + e_1)}{\alpha(x + e_1)}, \\ W(x + e_1 + e_2) &= W(x + e_2) \frac{\beta_1(x + e_2)}{\alpha(x + e_2)} = W(x) \frac{\beta_2(x)}{\alpha(x)} \frac{\beta_1(x + e_2)}{\alpha(x + e_2)}. \end{aligned}$$

Consequently, we conclude that

$$\frac{\beta_1(x)}{\alpha(x)} \frac{\beta_2(x + e_1)}{\alpha(x + e_1)} = \frac{\beta_2(x)}{\alpha(x)} \frac{\beta_1(x + e_2)}{\alpha(x + e_2)}$$

from which the equation (3.5) follows.

On the other hand, assume that (3.5) holds. We need to show that the system of equations (3.4) has a solution. We denote $\gamma_j := \beta_j/\alpha$. The condition (3.5) becomes

$$(3.7) \quad \frac{\gamma_1(x - e_2)}{\gamma_1(x)} = \frac{\gamma_2(x - e_1)}{\gamma_2(x)}.$$

Recall that the set V satisfies (2.3) and $(0, 0) \in V$. Clearly, multiplying γ_1 or γ_2 by a nonzero constant will not change the equation, so that we can assume that $\gamma_1(0) = 1$ and $\gamma_2(0) = 1$. For $x_1 \geq 0$ and $x_2 \geq 0$, we define a weight function W by

$$W(x) = \gamma_1(x - e_1)\gamma_1(x - 2e_1) \dots \gamma_1(0, x_2)\gamma_2(0, x_2 - 1)\gamma_2(0, x_2 - 2) \dots \gamma_2(0).$$

Using the equation (3.7), we then have

$$\begin{aligned} \frac{W(x + e_1)}{W(x)} &= \frac{\gamma_1(x)}{\gamma_1(0, x_2)} \frac{\gamma_2(1, x_2 - 1)}{\gamma_2(0, x_2 - 1)} \dots \frac{\gamma_2(1, 1)}{\gamma_2(0, 1)} \frac{\gamma_2(1, 0)}{\gamma_2(0, 0)} \\ &= \frac{\gamma_1(x)}{\gamma_1(0, x_2)} \frac{\gamma_1(0, x_2)}{\gamma_1(0, x_2 - 1)} \dots \frac{\gamma_1(0, 2)}{\gamma_1(0, 1)} \frac{\gamma_1(0, 1)}{\gamma_1(0, 0)} \\ &= \gamma_1(x). \end{aligned}$$

Furthermore, using the equation (3.10) again,

$$\begin{aligned} \frac{W(x + e_2)}{W(x)} &= \frac{\gamma_1(x_1 - 1, x_2 + 1)}{\gamma_1(x_1 - 1, x_2)} \frac{\gamma_1(x_1 - 2, x_2 + 1)}{\gamma_1(x_1 - 2, x_2)} \dots \frac{\gamma_1(0, x_2 + 1)}{\gamma_1(0, x_2)} \frac{\gamma_2(0, x_2)}{\gamma_2(0, 0)} \\ &= \frac{\gamma_2(x)}{\gamma_2(x - e_1)} \frac{\gamma_2(x - e_1)}{\gamma_2(x - 2e_1)} \dots \frac{\gamma_2(1, x_2)}{\gamma_2(0, x_2)} \frac{\gamma_2(0, x_2)}{\gamma_2(0, 0)} \\ &= \gamma_2(x). \end{aligned}$$

Consequently, the function W defined above is a solution for the equations (3.4). \square

3.2. Another notion of self-adjointness. We should like to point out that, just as in the definition of the standard difference equation, we can reverse the role of the forward and the backward differences and give another notion of self-adjointness.

Let $\tilde{a}_{i,j}$ be continuous functions of two variables. We could call a second order difference operator $\tilde{\mathcal{E}}$ self-adjoint if

$$(3.8) \quad \tilde{\mathcal{E}}u = \Delta_1 [\tilde{a}_{1,1}(x)\nabla_1 u + \tilde{a}_{1,2}(x)\nabla_2 u] + \Delta_2 [\tilde{a}_{2,1}(x)\nabla_1 u + \tilde{a}_{2,2}(x)\nabla_2 u].$$

For this notion of the self-adjointness, we can also determine the conditions under which $W\mathcal{D}$ is self-adjoint. Let us define

$$\begin{aligned} \tilde{\alpha}(x) &:= (A_{1,1}(x - e_1) + B_1(x))(A_{2,2}(x - e_2) + B_2(x)) - A_{2,1}(x - e_2)A_{1,2}(x - e_1), \\ \tilde{r}_1(x) &:= A_{1,1}(x) + A_{2,1}(x), \quad \text{and} \quad \tilde{r}_2(x) := A_{1,2}(x) + A_{2,2}(x). \end{aligned}$$

Following the proof that leads to the result for \mathcal{D} , we can prove the following:

Proposition 3.4. *Let \mathcal{D} be the difference operator in (2.5). There is a function W such that $W(x)\mathcal{D}$ is self-adjoint in the sense of (3.8) if and only if W satisfies*

$$(3.9) \quad \begin{aligned} WB_1 &= \nabla_1(WA_{1,1})(\cdot + e_1) + \nabla_2(WA_{1,2})(\cdot + e_1) \\ WB_2 &= \nabla_1(WA_{2,1})(\cdot + e_2) + \nabla_2(WA_{2,2})(\cdot + e_2). \end{aligned}$$

Furthermore, assume that $\tilde{\alpha}(x) \neq 0$; then there is a function W such that $W\mathcal{D}u$ is self-adjoint if and only if

$$\tilde{\alpha}(x - e_1)\tilde{r}_2(x)\tilde{r}_1(x - e_2) = \tilde{\alpha}(x - e_2)\tilde{r}_1(x)\tilde{r}_2(x - e_1).$$

3.3. Weight function and orthogonality. Given a weight function W defined on a lattice set V , we define

$$\mathcal{L}f = \sum_{x \in V} f(x)W(x).$$

For the difference operator given in \mathcal{D} , we define

$$(3.10) \quad \mathcal{A}_1 = A_{1,1}\nabla_1 + A_{1,2}\nabla_2 + B_1 \quad \text{and} \quad \mathcal{A}_2 = A_{2,1}\nabla_1 + A_{2,2}\nabla_2 + B_2.$$

From the definition of \mathcal{D} we have

$$(3.11) \quad \mathcal{D}u = \mathcal{A}_1\Delta_1u + \mathcal{A}_2\Delta_2u.$$

Let ∂V denote the boundary of the set V ; that is, ∂V is a subset of V such that if $x \in \partial V$ then at least one of the elements $x \pm e_i$, $i = 1, 2$, does not belong to V . In order that the difference equation makes sense on V , it needs to satisfy some condition on ∂V . For example, we can impose the conditions (3.12) below on the coefficients $A_{i,j}$.

Proposition 3.5. *If $W\mathcal{D}$ is self-adjoint and $A_{i,j}$ satisfy the conditions*

$$(3.12) \quad \begin{aligned} A_{1,1}(x) &= A_{2,1}(x) = 0, & x \in \partial V \text{ and } x + e_1 \in V \\ A_{1,2}(x) &= A_{2,2}(x) = 0, & x \in \partial V \text{ and } x + e_2 \in V, \end{aligned}$$

then, for any polynomial g ,

$$\mathcal{L}(\mathcal{A}_1g) = 0 \quad \text{and} \quad \mathcal{L}(\mathcal{A}_2g) = 0.$$

Proof. Let $\partial_1V = \{x \in \partial V : x + e_1 \in V\}$ and $\partial_2V = \{x \in \partial V : x + e_2 \in V\}$. Changing summation index and using the fact that $W(x + e_1) = 0$ if $x \in \partial V \setminus \partial_1V$, we obtain

$$\begin{aligned} \mathcal{L}(A_{1,1}\nabla_1g) &= \sum_{x \in V} W(x)A_{1,1}(x)\nabla_1g(x) \\ &= \sum_{x \in V} W(x)A_{1,1}(x)g(x) - \sum_{x+e_1 \in V} W(x+e_1)A_{1,1}(x+e_1)g(x) \\ &= - \sum_{x \in V} \Delta_1(WA_{1,1})(x)g(x) + \sum_{x \in \partial_1V} W(x)A_{1,1}(x)g(x). \end{aligned}$$

A similar equation holds for $\mathcal{L}(A_{1,2}\nabla_2g)$. Hence, by the assumption on $A_{i,j}$, we conclude that

$$\begin{aligned} \mathcal{L}(A_{1,1}\nabla_1g) &= - \sum_{x \in V} \Delta_1(WA_{1,1})(x)g(x), \\ \mathcal{L}(A_{1,2}\nabla_2g) &= - \sum_{x \in V} \Delta_2(WA_{1,2})(x)g(x). \end{aligned}$$

Since $W\mathcal{D}$ is self-adjoint if and only if (3.1) holds, by Proposition 3.2, it follows that

$$\begin{aligned} \mathcal{L}(\mathcal{A}_1g) &= \mathcal{L}(A_{1,1}\nabla_1g) + \mathcal{L}(A_{1,2}\nabla_2g) + \mathcal{L}(B_1g) \\ &= - \sum_{x \in V} [\Delta_1(WA_{1,1})(x) + \Delta_2(WA_{1,2})(x) - W(x)B_1(x)]g(x) = 0. \end{aligned}$$

The proof that $\mathcal{L}(\mathcal{A}_2g) = 0$ is similar. □

Definition 3.6. *The weight function W is consistent with the difference operator \mathcal{D} if, for all $g \in \Pi^2$,*

- (1) $\mathcal{L}(\mathcal{A}_1 g) = 0$ and $\mathcal{L}(\mathcal{A}_2 g) = 0$;
- (2) $\mathcal{L}(A_{1,2}g(\cdot - e_2)) = \mathcal{L}(A_{2,1}g(\cdot - e_1))$.

In particular, setting $g(x) = x_i$ shows that if W is consistent with \mathcal{D} then

$$\mathcal{L}(B_1) = 0 \quad \text{and} \quad \mathcal{L}(B_2) = 0.$$

By Proposition 3.5, if $W\mathcal{D}$ is self-adjoint and the coefficients of \mathcal{D} satisfies (3.12), then W is consistent with \mathcal{D} if the condition (2) in the above definition holds. We need the additional condition for the following result:

Theorem 3.7. *If W is consistent with the difference operator \mathcal{D} and the polynomials u and v satisfy $\mathcal{D}u = \lambda u$ and $\mathcal{D}v = \mu v$, then*

$$(\lambda - \mu)\mathcal{L}(uv) = 0.$$

Proof. By the definition of \mathcal{A}_1 and the product formula for Δ_j ,

$$\begin{aligned} \mathcal{A}_1(fg) &= A_{1,1}[f\nabla_1 g + g(\cdot - e_1)\nabla_1 f] + A_{1,2}[f\nabla_2 g + g(\cdot - e_2)\nabla_2 f] + B_1 fg \\ &= f\mathcal{A}_1 g + g(\cdot - e_1)A_{1,1}\nabla_1 f + g(\cdot - e_2)A_{1,2}\nabla_2 f. \end{aligned}$$

Applying \mathcal{L} to both sides and using the fact that W is consistent with \mathcal{D} , we conclude that

$$\mathcal{L}(f\mathcal{A}_1 g) = -\mathcal{L}(g(\cdot - e_1)A_{1,1}\nabla_1 f + g(\cdot - e_2)A_{1,2}\nabla_2 f).$$

Similarly, working with \mathcal{A}_2 , we have

$$\mathcal{L}(f\mathcal{A}_2 g) = -\mathcal{L}(g(\cdot - e_1)\nabla_1 A_{2,2}\nabla_1 f + g(\cdot - e_2)A_{2,2}\nabla_2 f).$$

From these two identities and the fact that $\Delta_j u(\cdot - e_j) = \nabla_j u$, it follows that

$$\begin{aligned} \mathcal{L}(v\mathcal{A}_1\Delta_1 u) &= -\mathcal{L}(\nabla_1 u A_{1,1}\nabla_1 v) - \mathcal{L}(\Delta_1 u(\cdot - e_2)A_{1,2}\nabla_2 v), \\ \mathcal{L}(v\mathcal{A}_2\Delta_2 u) &= -\mathcal{L}(\Delta_2 u(\cdot - e_1)A_{2,1}\nabla_1 v) - \mathcal{L}(\nabla_2 u A_{2,2}\nabla_2 v). \end{aligned}$$

Hence, by (3.11) and the fact that $\Delta_j u(\cdot - e_j) = \nabla_j u$, it follows that

$$\begin{aligned} (\lambda - \mu)\mathcal{L}(uv) &= \mathcal{L}(v\mathcal{A}_1\Delta_1 u) + \mathcal{L}(v\mathcal{A}_2\Delta_2 u) - \mathcal{L}(u\mathcal{A}_1\Delta_1 v) - \mathcal{L}(u\mathcal{A}_2\Delta_2 v) \\ &= -\mathcal{L}(\Delta_1 u(\cdot - e_2)A_{1,2}\nabla_2 v) - \mathcal{L}(\Delta_2 u(\cdot - e_1)A_{2,1}\nabla_1 v) \\ &\quad + \mathcal{L}(\Delta_1 v(\cdot - e_2)A_{1,2}\nabla_2 u) + \mathcal{L}(\Delta_2 v(\cdot - e_1)A_{2,1}\nabla_1 u) \\ &= -\mathcal{L}(A_{1,2}(\Delta_1 u\Delta_2 v)(\cdot - e_2)) - \mathcal{L}(A_{2,1}(\Delta_2 u\Delta_1 v)(\cdot - e_1)) \\ &\quad + \mathcal{L}(A_{1,2}(\Delta_1 v\Delta_2 u)(\cdot - e_2)) + \mathcal{L}(A_{2,1}(\Delta_2 v\Delta_1 u)(\cdot - e_1)) = 0 \end{aligned}$$

by the part (2) in the Definition 3.6 of the consistency. \square

Corollary 3.8. *If $W\mathcal{D}$ is self-adjoint and the coefficients of \mathcal{D} satisfies (3.12) and*

$$(3.13) \quad \mathcal{L}(A_{1,2}g(\cdot - e_2)) = \mathcal{L}(A_{2,1}g(\cdot - e_1)), \quad g \in \Pi^2,$$

then the equation $\mathcal{D}u = \lambda_k u$ has polynomial solutions that are orthogonal with respect to W on V .

3.4. Condition for consistency. Let us look at the conditions for the consistency closely. Recall that the Type A set $V = \{(x_1, x_2) : 0 \leq x_1 \leq M, 0 \leq x_2 \leq N\}$ and the Type B set $V = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, 0 \leq x_1 + x_2 \leq N\}$. If V is of Type A or Type B, the condition (3.12) means

$$(3.14) \quad A_{1,1}(0, x_2) = A_{2,1}(0, x_2) = 0, \quad \text{and} \quad A_{1,2}(x_1, 0) = A_{2,2}(x_1, 0) = 0.$$

Since $A_{1,1}$ and $A_{2,1}$ are coefficients of $\Delta_1 \nabla_1$ and $\Delta_2 \nabla_1$, both involve ∇_1 , and $A_{2,1}$ and $A_{2,2}$ are coefficients of $\Delta_1 \nabla_2$ and $\Delta_2 \nabla_2$, both involve ∇_2 , the conditions in (3.14) appear to be reasonable assumptions.

Recall that $\partial_1 V = \{x \in \partial V : x + e_1 \in V\}$ and $\partial_2 V = \{x \in \partial V : x + e_2 \in V\}$.

Proposition 3.9. *Assume that $A_{1,2} - A_{2,1} \in \Pi_1^2$. If \mathcal{D} is admissible and (3.12) holds, then $\partial_1 V$ is a subset of $\{x : x_1 = 0\}$ and $\partial_2 V$ is a subset of $\{x : x_2 = 0\}$. In particular, (3.12) becomes (3.14).*

Proof. The condition (3.12) means that the pair $A_{1,1}(x)$ and $A_{2,1}(x)$, and the pair $A_{1,2}(x)$ and $A_{2,2}(x)$, agrees on the set ∂V_1 . Recall that both $A_{1,1}$ and $A_{2,1}$ are quadratic polynomials and they are of the form (2.7). If ∂V_1 contains more than 4 points, then the classical Bezout theorem on the algebraic curves will imply that $A_{1,1}$ and $A_{2,1}$ has a common factor. Since $A_{1,1}(x) = ax_1^2 + \pi_{1,1}(x)$ and $A_{2,1}(x) = ax_1x_2 + \pi_{2,1}(x)$, where $\pi_{i,j}$ are polynomials of degree 1, the common factor has to be a linear polynomial, which means that both polynomials are product of linear factors. Thus, they must be of the form

$$A_{1,1}(x) = ax_1(x_1 + b_1) \quad \text{and} \quad A_{2,1}(x) = ax_1(x_2 + b_2).$$

Consequently, $A_{1,1}(0, x_2) = 0$ and $A_{2,1}(0, x_1) = 0$ become the only possibility. Thus, ∂V_1 is a subset of $\{x : x_1 = 0\}$.

The same consideration applies to the pair $A_{1,2}(x)$ and $A_{2,2}(x)$ and shows that ∂V_2 is a subset of $\{x : x_2 = 0\}$. \square

Corollary 3.10. *Assume that $A_{1,2} - A_{2,1} \in \Pi_1^2$. If $W\mathcal{D}$ is self-adjoint and the weight function W is consistent with \mathcal{D} , then*

$$(3.15) \quad \begin{aligned} A_{1,1}(x) &= ax_1(c_1 - x_1), & A_{1,2}(x) &= ax_2(c_2 - x_1), & B_1(x) &= bx + d_1; \\ A_{2,1}(x) &= ax_1(c_3 - x_2), & A_{2,2}(x) &= ax_2(c_4 - x_2), & B_2(x) &= bx + d_2. \end{aligned}$$

Notice that (3.14) only imposes conditions on $A_{i,j}$ on part of ∂V . If V is finite, there should also be conditions on the set $\partial V \setminus (\partial_1 V \cup \partial_2 V)$. The equation (3.13) imposes such a condition, which is often fulfilled if the weight function W vanishes on the set $\partial V \setminus (\partial_1 V \cup \partial_2 V)$.

The property that $\partial_1 V$ and $\partial_2 V$ are subsets of $\{x : x_2 = 0\}$ and $\{x : x_1 = 0\}$, respectively, is fulfilled by the Type A and the Type B sets. In fact, they appear to be the only sets V that satisfy this property in connection with the difference equation.

4. DIFFERENCE EQUATIONS AND ORTHOGONAL POLYNOMIALS

4.1. Preliminary. The results in the previous section provides us with a method to identify difference equations that have orthogonal polynomials as solutions. The method consists of the following steps.

Step 1. Among all admissible difference operators that satisfy (3.15), find \mathcal{D} that satisfies (3.5).

Step 2. Using (3.6) to identify the weight function that satisfies (3.1).

Step 3. Verifying that the condition (3.13) holds.

We will follow the steps outlined above to do a case by case study. The classical discrete orthogonal polynomials of one variable will appear in the study, so we list them below (see [11] or [7]):

Hahn polynomials $u = Q_n(x; \alpha, \beta, N)$:

$$t(N + \beta + 1 - t)\Delta\nabla u + (N(\alpha + 1) - (\alpha + \beta + 2)t)\Delta u = -n(n + \alpha + \beta + 1)u,$$

where $\alpha, \beta \geq 0$ and $N \in \mathbb{N}$. They are orthogonal with respect to the weight function $\binom{\alpha+t}{\alpha}\binom{\beta+N-t}{\beta}$ on $V = \{0, 1, \dots, N\}$.

Meixner polynomials $u = M_n(t; \beta, c)$:

$$t\Delta\nabla u + [c(t + \beta) - t]\Delta u = n(c - 1)u$$

where $\beta > 0$ and $0 < c < 1$. They are orthogonal with respect to the weight function $\frac{(\beta)_x}{x!}c^x$ on $V = \mathbb{N}_0$.

Krawtchouk polynomials $u = K_n(t; p, N)$:

$$t(1 - p)\Delta\nabla u + [p(N - t) - t(1 - p)]\Delta u = -nu$$

where $p > 0$ and $N \in \mathbb{N}_0$. They are orthogonal with respect to the weight function $\binom{N}{t}p^t(1 - p)^{N-t}$ on $V = \{1, 2, \dots, N\}$.

Charlier polynomials $u = C_n(t; a)$:

$$t\Delta\nabla u + (a - t)\Delta u = -nu$$

where $a > 0$. They are orthogonal with respect to the weight function $a^t/t!$ on $V = \mathbb{N}_0$.

4.2. Difference equations with orthogonal polynomial solutions. We will now follow the steps outlined to identify the second order difference equations that have orthogonal polynomial as solutions. For this we assume that \mathcal{D} is admissible and the coefficients $A_{i,j}$ of \mathcal{D} satisfy (3.15).

The equation (3.5) is a nonlinear equation, we solve it with the help of a computer algebra system.

Case 1: $a \neq 0$. We can assume that $a = -1$. Then $A_{i,j}$ and B_i must take the form a

$$\begin{aligned} A_{1,1}(x) &= x_1(c_1 - x_1), & A_{1,2}(x) &= x_2(c_2 - x_1), & B_1(x) &= bx + d_1; \\ A_{2,1}(x) &= x_1(c_3 - x_2), & A_{2,2}(x) &= x_2(c_4 - x_2), & B_2(x) &= bx + d_2. \end{aligned}$$

With the $A_{i,j}$ and B_i given as above, solving the equation (3.5) (with the help of a computer algebra system) leads to essentially two solutions.

Case 1 (i): One solution is

$$d_2 = -(b + c_1 - c_2)c_3, \quad d_1 = c_2(-b - c_1 + c_2), \quad c_4 = c_1 - c_2 + c_3,$$

so that, by solving (3.6), we get

$$W(x) = \frac{\Gamma(-c_2 + x_1)\Gamma(-c_3 + x_2)\Gamma(c_1 + c_3 - x_1 - x_2)}{\Gamma(1 + x_1)\Gamma(1 + x_2)\Gamma(1 + b + c_1 - c_2 - x_1 - x_2)}.$$

It is easy to verify that (3.13) holds for $V = \{x : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq N\}$.

This turns out to be exactly the Hahn polynomials of two variables discussed in [4]. Indeed, setting $c_2 = -(\sigma_1 + 1)$, $c_3 = -(\sigma_2 + 1)$, $c_1 = N + \sigma_2 + \sigma_3 + 2$ and $b = -3 - \sigma_1 - \sigma_2 - \sigma_3$ shows that the difference equation is the one given below:

Hahn Polynomials.

$$(4.1) \quad \begin{aligned} & x_1(N - x_1 + \sigma_2 + \sigma_3 + 2)\Delta_1\nabla_1 u - x_2(x_1 + \sigma_1 + 1)\Delta_1\nabla_2 u \\ & - x_1(x_2 + \sigma_2 + 1)\Delta_2\nabla_1 u + x_2(N - x_2 + \sigma_1 + \sigma_3 + 2)\Delta_2\nabla_2 u \\ & + [(N - x_1)(\sigma_1 + 1) - x_1(\sigma_2 + \sigma_3 + 2)]\Delta_1 u \\ & + [(N - x_2)(\sigma_2 + 1) - x_2(\sigma_1 + \sigma_3 + 2)]\Delta_2 u = -n(n + |\sigma| + 2)u, \end{aligned}$$

where $|\sigma| = \sigma_1 + \sigma_2 + \sigma_3$. The weight function W is defined on $V = \{x : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq N\}$ by

$$W_\sigma(x) = \binom{x_1 + \sigma_1}{\sigma_1} \binom{x_2 + \sigma_2}{\sigma_2} \binom{N - x_1 - x_2 + \sigma_3}{\sigma_3}.$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; \sigma)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Hahn polynomials by

$$\phi_{l,m}(x; \sigma) = Q_l(x_1; \sigma_1, \sigma_2 + \sigma_3 + 2m + 1, N - l)(-N + x)_m Q_m(x_2; \sigma_2, \sigma_3, N - x_1).$$

Case 1 (ii): The other solution is

$$d_1 = c_2(-b - c_1 + c_2), \quad d_2 = -(1 + b)c_3, \quad c_4 = 1 + c_3,$$

so that, by solving (3.6), we get

$$W(x) = \frac{\Gamma(-c_2 + x_1)\Gamma(-1 - b + c_2 + x_2)}{\Gamma(1 + x_1)\Gamma(1 + x_2)}.$$

The consistent condition (3.13) with $g(x) = x_i$ shows that $b = c_2 + c_3 - 1$. Setting $c_2 = -\kappa_1 - 1$, $c_3 = -\kappa_2 - 1$ and $c_1 = M + \kappa_2 + 2$ shows that the difference equation is of the same form as the equation (4.1) for the Hahn polynomials of two variables with $\sigma_1 = -\kappa_1$, $\sigma_2 = -\kappa_2$ and $N = -2 - \kappa_1 - \kappa_2$. Since M has to be a positive integer, this shows that κ_1 and κ_2 need to be negative. However, in this case,

$$W(x) = \frac{\Gamma(1 + \kappa_1 + x_1)\Gamma(1 + \kappa_2 + x_2)}{\Gamma(1 + x_1)\Gamma(1 + x_2)}$$

has singularities at $x_1 = -\kappa_1 - 1$ or $x_2 = -\kappa_2 - 1$. Thus, this is not a proper solution.

Remark. The computer algebra system that we used produced several solutions for (3.5), upon close examination, however, we have (4.1) as essentially the only solution. The other solutions either turn out to be special cases of (4.1) or like the case (ii) given above. Even though the case (ii) does not lead to a proper solution, we have included the case to indicate how the method works.

Case 2: $a = 0$. Assume that \mathcal{D} is admissible and (3.14) holds. Then $A_{i,j}$ and B_i must take the form

$$\begin{aligned} A_{1,1}(x) &= a_1 x_1, & A_{1,2}(x) &= a_2 x_2, & B_1(x) &= b x + d_1; \\ A_{2,1}(x) &= a_3 x_1, & A_{2,2}(x) &= a_4 x_2, & B_2(x) &= b x + d_2. \end{aligned}$$

With these $A_{i,j}$ and B_i , solving the equation (3.5) (with the help of a computer algebra system) shows that there are several solutions.

Case 2 (i): $a_2 \neq 0$ and $a_3 \neq 0$. In this case we get

$$d_1 = a_2 d_2 / a_3, \quad a_1 = a_2 - b, \quad a_4 = a_3 - b$$

and, by solving (3.6),

$$W(x) = \frac{(-a_2)^x (-a_3)^y (a_2 + a_3 - b)^{N-x-y}}{\Gamma(1 - d_2/a_3 - x - y) \Gamma(1+x) \Gamma(1+y)}.$$

It is easy to verify that the condition (3.13) holds for $V = \{x : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq N\}$. This turns out to be exactly the Krawtchuk polynomials of two variables on V . Indeed, setting $b = -1$, $a_2 = -p_1$, $a_3 = -p_2$ and $d_2 = a_3(1 - N)$ shows that the difference equation is the one given below:

Krawtchuk Polynomials.

$$(4.2) \quad (1 - p_1)x_1 \Delta_1 \nabla_1 u - p_1 x_2 \Delta_1 \nabla_2 u - p_2 x_1 \Delta_2 \nabla_1 u + (1 - p_2)x_2 \Delta_2 \nabla_2 u \\ + [p_1(N - x_1) - (1 - p_1)x_1] \Delta_1 u + [p_2(N - x_2) - (1 - p_2)x_2] \Delta_2 u \\ = -nu.$$

The weight function W is defined on $V = \{x : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq N\}$ by

$$W_p(x) = \frac{N!}{x!y!(N-x-y)!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{N-x_1-x_2}.$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; p, N)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Krawtchuk polynomials by

$$\phi_{l,m}(x; p, N) = K_l(x; p_1, N - m)(-N + x)_m K_m(y; p_2/(1 - p_1), N - x).$$

These polynomials have appeared in [13] but the difference equation was not given there.

Case 2 (ii): $a_2 = 0$ and $a_3 = 0$. In this case we have several solutions, depending on the signs of various coefficients. These correspond to the product type orthogonal polynomials, which we list below.

Product Meixner polynomials:

$$(4.3) \quad \frac{1}{c_1 - 1} x_1 \Delta_1 \nabla_1 u + \frac{1}{c_2 - 1} x_2 \Delta_2 \nabla_2 u \\ + \frac{1}{c_1 - 1} [c_1(x_1 + \beta_1) - x_1] \Delta_1 u + \frac{1}{c_2 - 1} [c_2(x_2 + \beta_2) - x_2] \Delta_2 u = nu.$$

The weight function W is defined on $V = \mathbb{N}_0^2$ by

$$W_{b,\beta}(x) = \frac{(\beta_1)_{x_1}}{x_1!} c_1^{x_1} \frac{(\beta_2)_{x_2}}{x_2!} c_2^{x_2}.$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; \beta, c)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Meixner polynomials by

$$\phi_n(x; \beta, c) = M_l(x_1; \beta_1, c_1) M_m(x_2; \beta_2, c_2).$$

Product Meixner-Krawtchuk polynomials:

$$(4.4) \quad \frac{1}{c - 1} x_1 \Delta_1 \nabla_1 u - (1 - p)x_2 \Delta_2 \nabla_2 u \\ + \frac{1}{c - 1} [c(x_1 + \beta) - x_1] \Delta_1 u - [p(N - x_2) - (1 - p)x_2] \Delta_2 u = nu.$$

The weight function W is defined on $V = \{(i, j) : i \in \mathbb{N}_0, 0 \leq j \leq N\}$ by

$$W_{c,\beta,p}(x) = \frac{(\beta)_{x_1}}{x_1!} c^{x_1} \binom{N}{x_2} p^{x_2} (1-p_2)^{N-x_2}.$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; \beta, c, p, N)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Meixner and Krawtchuk polynomials by

$$\phi_n(x; \beta, c, p, N) = M_l(x_1; \beta, c) K_m(x_2; p, N).$$

Product Meixner-Charlier polynomials:

$$(4.5) \quad \begin{aligned} & \frac{1}{c-1} x_1 \Delta_1 \nabla_1 u - x_2 \Delta_2 \nabla_2 u \\ & + \frac{1}{c-1} [c(x_1 + \beta) - x_1] \Delta_1 u - (a - x_2) \Delta_2 u = nu. \end{aligned}$$

The weight function W is defined on $V = \mathbb{N}_0^2$ by

$$W_{c,\beta,a}(x) = \frac{(\beta)_{x_1}}{x_1!} c^{x_1} \frac{(a)_{x_2}}{x_2!}.$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; \beta, c, a)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Meixner and Charlier polynomials by

$$\phi_n(x; \beta, c, a) = M_l(x_1; \beta, c) C_m(x_2; a).$$

Product Krawtchuk polynomials:

$$(4.6) \quad \begin{aligned} & (1-p_1)x_1 \Delta_1 \nabla_1 u + (1-p_2)x_2 \Delta_2 \nabla_2 u \\ & + [p_1(N_1 - x_1) - (1-p_1)x_1] \Delta_1 u + [p_2(N_2 - x_2) - (1-p_2)x_2] \Delta_2 u = -nu. \end{aligned}$$

The weight function W is defined on $V = \{(i, j) : 0 \leq i \leq N_1, 0 \leq j \leq N_2\}$ by

$$W_p(x) = \binom{N_1}{x_1} p_1^{x_1} (1-p_1)^{N_1-x_1} \binom{N_2}{x_2} p_2^{x_2} (1-p_2)^{N_2-x_2}.$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; p, N)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Krawtchuk polynomials by

$$\phi_n(x; p, N) = K_l(x_1; p_1, N_1) K_m(x_2; p_2, N_2).$$

Product Krawtchuk-Charlier polynomials:

$$(4.7) \quad \begin{aligned} & (1-p)x_1 \Delta_1 \nabla_1 u + x_2 \Delta_2 \nabla_2 u \\ & + [p(N - x_1) - (1-p)x_1] \Delta_1 u + (a - x_2) \Delta_2 u = -nu. \end{aligned}$$

The weight function W is defined on $V = \{(i, j) : 0 \leq i \leq N, j \in \mathbb{N}_0\}$ by

$$W_{p,a}(x) = \binom{N, x_1}{p}^{x_1} (1-p)^{N-x_1} \frac{(a)_{x_2}}{x_2!}$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; a, p, N)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Krawtchuk and Charlier polynomials by

$$\phi_n(x; a, p, N) = K_l(x_1; p, N) C_m(x_2; a).$$

Product Charlier polynomials:

$$(4.8) \quad x_1 \Delta_1 \nabla_1 u + x_2 \Delta_2 \nabla_2 u + (a_1 - x_1) \Delta_1 u + (a_2 - x_2) \Delta_2 u = -nu.$$

The weight function W is defined on \mathbb{N}_0^2 by

$$W_a(x) = \frac{a_1^{x_1} a_2^{x_2}}{x_1! x_2!}$$

For each n , the equation has solutions $\phi_{l,m}(\cdot; a)$, $m = n - l$ and $0 \leq l \leq n$, given in terms of Charlier polynomials by

$$\phi_n(x; a) = C_l(x_1; a_1) C_m(x_2; a_2).$$

Case 2 (iii), other possibilities. There is no solution if one of the a_2 or a_3 is zero and the other one is not. However, if we set $a_4 = 0$, then solving the equation (3.5) leads to

$$d_1 = -(a_1 + b)d_2/b, \quad a_2 = a_1 + b, \quad a_3 = b.$$

Solving the equation (3.6) leads to the weight function

$$W(x) = b^y (a_1 + b)^{N-y} \Gamma(d_2/b + x + y) / (x! y!).$$

Since W does not have enough decay, the set V needs to be bounded to keep the linear functional \mathcal{L} defined by W well-defined on polynomials. However, we do not see a way to choose V such that W vanishes on the boundary $\partial V \setminus (\partial_1 V \cap \partial_1 V)$. In other words, we do not see a way to choose V such that the condition (3.13) can be satisfied.

4.3. Final Remarks. Under the assumption that $A_{i,j}$ are of the form (3.15), we found eight difference equations that have orthogonal polynomials as solutions. These are the Hahn and the Krawtchuk polynomials on Type A set V , and product type polynomials on Type B set V : product Meixner, product Krawtchuk, and product Charlier polynomials, as well as product Meixner-Krawtchuk, Meixner-Charlier, and Krawtchuk-Charlier polynomials. With the restrictions that we put on the difference equations (that is, the conditions in Corollary 3.10), these appear to be the only possible solutions.

Let us mention that another attempt of characterizing orthogonal polynomials of two variables satisfying second order difference equations, based on a matrix approach, is currently being undertaken by a group in University of Granada ([10]).

As pointed out in the introduction, if u satisfies a difference equation then the function $u_h(x) := u(h_1 x_1, h_2 x_2)$ satisfies a similar difference equation which becomes in limit, as $h \rightarrow 0$, the second order partial differential equation. The second order partial differential equations that have orthogonal polynomials as solutions are characterized in [8]. It is known that there are nine such equations, whose solutions correspond to product type orthogonal polynomials of two variables, as well as two other type, one is the Jacobi type orthogonal polynomials on the triangle $T = \{(x, y) : x \geq 0, y \geq 0, 0 \leq x + y \leq 1\}$ and the other is the orthogonal polynomials on the disk $B = \{(x, y) : x^2 + y^2 \leq 1\}$. The discrete Hahn polynomials of two variables on Type A set appear to become the Jacobi type polynomials on the triangle. However, as far as we can tell, there is no discrete analog for the orthogonal polynomials on the disk.

A natural question is if there are other difference equations of the form (1.3) that have orthogonal polynomials as solutions. Notice that our results are obtained under various assumptions on \mathcal{D} . For example, we assume that the quadratic parts of $A_{1,2}$ and $A_{2,1}$ are equal, that is, $A_{1,2} - A_{2,1} \in \Pi_1^2$. Furthermore, we establish the orthogonality by requiring that \mathcal{D} is self-adjoint and W is consistent with \mathcal{D} .

These assumptions are sufficient but may not be necessary. Still, assuming that $A_{1,2} - A_{2,1} \in \Pi_1^2$, it seems to us that likely no other nontrivial difference equations can have orthogonal polynomials as solutions.

REFERENCES

- [1] W. A. Al-Salam, Characterization theorems for orthogonal polynomials, in *Orthogonal polynomials (Columbus, OH, 1989)*, p. 1–24, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 294, Kluwer Acad. Publ., Dordrecht, 1990.
- [2] H. Berens, H. Schmid and Yuan Xu, On two-dimensional definite orthogonal systems and on lower bound for the number of associated cubature formulas, *SIAM J. Math. Anal.* **26** (1995), 468–487.
- [3] C. F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, Cambridge Univ. Press, 2001.
- [4] S. Karlin and J. McGregor, Linear growth models with many types and multidimensional Hahn polynomials, in *Theory and applications of special functions*, 261–288, ed. R. A. Askey, Academic Press, New York, 1975.
- [5] Y. J. Kim, K. H. Kwon and J. K. Lee, Partial differential equations having orthogonal polynomial solutions, *J. Comput. Appl. Math.* **99** (1998), 239–253.
- [6] K. H. Kwon, J. K. Lee and L. Littlejohn, Orthogonal polynomial eigenfunctions of second order partial differential equations, *Trans. Amer. Math. Soc.*, **353** (2001), 3629–3647.
- [7] R. Koekoek and R. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analog*, vol. 98-17 of *Reports of the faculty of Technical Math. and Infor.*, Delft, Netherlands, 1998. Available at <http://aw.twi.tudelft.nl/~koekoek/research.html>
- [8] H. L. Krall and I. M. Sheffer Orthogonal polynomials in two variables, *Ann. Mat. Pura Appl.* (4) **76** (1967), 325–376.
- [9] L. L. Littlejohn, Orthogonal polynomial solutions to ordinary and partial differential equations, in *Orthogonal polynomials and their applications (Segovia, 1986)*, *Lecture Notes in Math.* **1329**, 98–124. Springer, Berlin, 1988.
- [10] M. A. de Morales, L. Gernández, T. E. Pérez and M. A. Piñar, Bivariate orthogonal polynomials satisfying second order difference equations, *abstract of 5th International Conference on Functional Analysis and Approximation Theory*, Maratea, Italy, June 16-23, 2004.
- [11] A.F. Nikiforov, S.K. Suslov and V.B. Uvarov, *Classical orthogonal polynomials of a discrete variable*, Springer-Verlag, Berlin, 1991.
- [12] P. K. Suetin, *Orthogonal polynomials in two variables*, translated from the 1988 Russian original by E. V. Pankratiev, Gordon and Breach, Amsterdam, 1999.
- [13] M. V. Tratnik, Multivariable Meixner, Krawtchouk, and Meixner-Pollaczek polynomials, *J. Math. Phys.* **30** (1989), 2740–2749.
- [14] M. V. Tratnik, Some multivariable orthogonal polynomials of the Askey tableau - discrete families. *J. Math. Phys.* **32** (1991), 2337–2342.
- [15] Yuan Xu On discrete orthogonal polynomials in several variables, *Advances in Applied Math.*, to appear.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403-1222.
E-mail address: `yuan@math.uoregon.edu`